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Abstract: We consider weak eulerian evolution of domains through the convection of a measurable set by a nonsmooth vector field V. We introduce the concept of tubes by "product space" and we show a closure result leading to existence results in non cylindrical shape evolution and a new shape metric using the tube geodesic

Key-words: convection, tube, shape control in non cylindrical heat equation, new shape metric

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Controle par la géométrie et métrique de domaine par geodesique de tubes

Résumé : On considère l'évolution faible de domaine via la convection d'une partie mesurable par un champ V peu régulier. On introduit le concept de tube "par espace produit" et l'on établit un résultat de fermeture conduisant à des résultats d'existence en contrôle par le domaine mobile et à une nouvelle métrique de domaines utilisant les géodésiques de tubes

Mots-clés : convection, tube, contrôle par le domaine dans la chaleur non cylindrique, nouvelle topologie sur les domaines par géodésiques de tubes

Contents

1 Introduction

The weak convection of measurable sets ([21], [20], [12]) has been introduced in relation with the shape differential equation and related topics such as variational formulation of Euler equation.

1.1 Weak convection of domain

Possible topological changes were already considered in [18], [19], ... as a nonsmooth extension of the "Speed Method" introduced in [15], [16], [17]i,...[12],[3] to describe a large evolution of geometrical domain (or shape or image). The *transverse variation* analysis led to derivative (with respect to the speed vector field) of functional associated to the evolution *tube*. We proposed eulerian variational formulation for several classical problems such as incompressible euler flow (in [12], [20], minimal curves, elastic shells.... in[21]. Those derivatives turn to be governed by a *geometrical adjoint state* λ which is backward and is obtained with the use of the so-called *tranvserse field* Z introduced in [19]. The shape differential equation introduced in 1976 ([15]), extended to the level set approach (whose speed vector approach was contained in the free boundary modeling in 1980 [17]ii)should now be developed using toghether the weak evolution and the so called "topological derivative". We enlarge that approach to variational problem related to the analysis and Tubes evolution . We focus on the compacity results and we proposed three "families" of "viscosity constraints" on the vector fields V leading to existence results connected to the "parabolic version" of the compactness of the inclusion mapping from bounded variation functions in integrale functions . The last one is based on the use of perimeter and "Density Perimeter" properties ([7], [8]).

1.2 Tube by product spaces

The present paper is concerned with a more complete analysis of the "tube by product spaces" approach introduced in [21]. A tube is a couple (ζ, V) satisfying the convection problem 13. when V is smooth , the characteristic function can be deduced from the initial domain as $\zeta(t) = \chi_{\Omega_0} o T_t(V)^{-1}$, where $T_t(V)$ is the flow mapping associated to V. The main point is the result concerning the *closure* of the family of such tubes under perimeter constraints. Using a specific choice of control functional in the heat equation we make use of weak compacity associated with the closure in order to derive existence result for both weak solution of the heat equation and for the non smooth optimal tube (in which the heat equation solution turns to exists).

1.3 Application to Shape distance by Tubes.

The tube induces a characteristic function which is time continuous, namely $\zeta \in C^0([0,\tau], L^2(D))$ so that we say that two domains Ω_1 and Ω_2 are *tube-connected* (or simply connected) if there exists a tube (ζ, V) such that $\zeta(0) = \chi_{\Omega_1}$, $\zeta(1) = \chi_{\Omega_2}$ (here the final time can be choosen as $\tau = 1$). The new distance is associated to the minimization with respect to such tubes (ζ, V) of a functional involving perimeter terms and norms in L^1 of both V and its divergence field.

2 Smooth (or Classical) Tubes

Being given for each t as smooth domain Ω_t, we consider the non cylindrical evolution domain

$$Q = \bigcup_{0<t<\tau} \{t\} \times \Omega_t \tag{1}$$

With smooth lateral boundary

$$\Sigma = \bigcup_{0<t<\tau} \{t\} \times \Gamma_t \tag{2}$$

We denote by ν the unitary normal field on Σ, it can be written as :

$$\nu(t,x) = \frac{1}{\sqrt{(1+v(t,x)^2)}}(-v(t,x)\,,\, n_t(x)\,) \tag{3}$$

We call v the "normal speed of the boundary" as, following [16], any smooth vector field V whose flow $T_t(V)$ builts that Q (i.e. such that $\Omega_t = T_t(V)(\Omega_0)$) verifies $< V(t), n_t >= v(t)$ on $\Gamma_t := \partial\Omega_t$. Conversely if $v =< V, n_t >$ then the field V built Q. When Q is smooth enough it can always be built by a gradient: there exists $V = \nabla A$ which builts Q (see[21]).

3 Heat Equation with Insulated Boundary

We consider the non cylindrical situation: the boundary Σ is insulated or adiabatic. As the domains moves it is not the usual neumann boundary condition but the one described bellow.

3.1 The Strong Formulation

Non cylindrical evolution problems, such as Navier Stokes equation for moving boundaries in a fluid (see [11])is a chalenging optimal control issue. In the case of linear problems we deal with easier situations. Nethertheless a difficult issue is that we need to handle such problem with non smooth geometry. The study of non cylindrical heat equation is a old story. Far from being exhaustive here let us quote the works by P. Acquistapace [1], and recentely in [4]. In these works the boundary of the moving domain should be smooth enough. The obvious technic was based on the transport into a cylindrical problem which, in term of abstract setting, leads to a dynamical system with a non autonomeous operator with a moving domain. Here we revisit that analysis in the scope of the Optimal Control of the moving domain Ω_t. As classicaly in shape analysis, the control parameter will be the speed vector field $V(t,x)$ whose flow mapping $T_t(V)$ builts the non cylindrical evolution domain $Q_V = \cup_{0<t<\tau}\{t\}\times\Omega_t$. Let $V \in C^0([0,\tau], C^1(D,R^N))$ with $V.n = 0$ on ∂D, the moving domain is $\Omega_t := T_t(V)(\Omega_0)$ and its characterisitic function is $\zeta = \zeta_0 o T_t(V)^{-1}$. We consider the unique solution u to the parabolic problem:

$$\frac{\partial}{\partial t}u - \Delta u = 0 \text{ in } Q_V, \ \ \frac{\partial}{\partial n_t}u + v\ u = 0 \ \text{ on the moving boundary } \Gamma_t, \ \ u(0) = u_0 \tag{4}$$

This b.c. cannot be written as $\frac{\partial}{\partial\nu}u = 0$ on the lateral time-space boundary Σ.

3.2 The weak formulation

is the following one:

$$\forall\psi \in C^1([0,\tau]\times R^n) \text{ with } \psi(\tau) = 0, \int_0^\tau\int_{\Omega_t}(-u\frac{\partial}{\partial t}\psi + \nabla u.\nabla\psi)dtdx = \int_{\Omega_0}\psi(0)(x)\,dx \tag{5}$$

Introducing $U(t,x) = u(t)oT_t(V)(x)$ the transported solution on the cylindrical domain, we get U as solution to the parabolic boundary value problem:

$$U_t + U(ln\,J)_t - J^{-1}div(U\,J\,DT^{-1}.V(t)oT_t(V)) - J^{-1}\,div(J\,DT^{-1}.(DT^*)^{-1}.\nabla U) = 0 \tag{6}$$

with the boundary condition

$$< DT_t(V)^{-1}.(DT_t(V)^{-1})^*.\nabla U, n > + < DT_t(V)^{-1}.V(t)oT_t(V), n > U = 0 \tag{7}$$

3.3 Optimal Control Problem

Given some element $U_d \in L^2(D)$ and $\sigma > 0$, we introduce functional in the following form

$$j(V) = 1/2\int_0^\tau\int_D \zeta(\,(u-U_d)^2 + |\nabla u|^2)dxdt + \ \sigma/2\int_0^\tau |V(t)|^2\,dt \tag{8}$$

For some ("strong enough") norm on the vector field $V(t)$. Then that functional should be minimized with respect to the vector field V. If the norm on the field is in the form $|V(t)| = ||V(t)||_{H^s(D,R^N)}$, with s large enough, that functional would reach its minimum on $E_s = L^2(0,\tau,H^s(D,R^N))$. The objective of that work is to understand in which framework we could extend that type of result to non smooth tubes involving possible topological changes in the evolution domain, such as creation of holes, changing the number of connected componants. . The basic classical idea being to characterise the tube by the convected characteristic function $\zeta(t)$ of the moving domain. The convection equation by the vector field V need not V to be smooth (say Lipschitz continuous). The arising question in that analysis is to caracterise the "minimal" regularity on the vector V which enables the convection analysis. From the work of Di Perna and P.L. Lions [10] , P.L. Lions [14], the space $W^{1,1}(D, R^N)$ furnishes a solution through a very thin analysis of the flow mapping defined almost every where. Recentely those results are improved by L. Ambrosio [2] who , using the finite rank theorem of Alberti, solves the convection equation and associate measure transportation problem, with vector field ranging in the $BV(D, R^N)$ linear space (with also some uniform boundedness).In these situations it is very difficult to treat the variation problem of the functional $j(V)$ as the involved norms induce complicated weak topologies , weak compactness and lower semicontinuity. We adopt here a different view point : taking *advantage* of the minimization problem we will add to the functional a term in the form:

$$\sigma||P(\partial\Omega_t)||_{BV(0,\tau)} \tag{9}$$

where P will be "a perimeter" (namely here the usual BV perimeter in D or the Density Perimeter P_γ introduced in [8],[7]) but could be also some Sobolev norm of the oriented distance function in a tubular neighboohoud around the boundary, see [6]. We will consider minimizing sequences V_n in the lipchitz continous class E and from the previous additive term we will get some perimeter boundedness which will be enough to "forece" both tubes Q_{V_n} and V_n to respectively converge to limit element Q, V with $Q = Q_V$ in the convection meaning that we shall make precise in the next section. Then we solve the minimization problem on some "natural" closure of the familly of weak tubes (Q_n, V_n) that we shall denote $\mathcal{T}^{p,\infty}_{\Omega_0}$, see 28. Of course for such very non smooth tubes Q there is no existence results for the solution u to the heat problem. Nethertheless trough its weak formulation and the boundedness of u_n which derives from the minimization of $j(V_n)$ we shall derive that u_n weakly converges to some limiting element u which turns to be a weak solution to the heat problem under consideration in the non smooth tube Q. In that sense we claim that (under the had hoc choice of the functional j in energy form) the optimal control of the solution leads to the existence result. We hope that simple exemple could open the technic to more complicated situation as control of moving shapes in incompressible viscous fluid. That last point will require the perimeter term to be more "reach" than the BV one. In other words, the Cacciopoli sets seem not adapted for the PDE analysis as they are not open (nor necessarely quasi open) subsets.It is the reason for which we propose to make use of $P_\gamma(\partial\Omega_t)$ which permit to fully characgterize the limiting element u as a solution to the weak form of theheat problem in the limiting non smooth tube Q.

3.4 Optimization Problem

We consider the minimization problem

$$Inf\ \{\ j(V,\zeta)\ \ |\ \ (V,\zeta)\in\mathcal{T}_\infty\ \} \tag{10}$$

4 Domain Convection

We consider a bounded open domain D in R^N and smooth vector field $V \in C^0([0,\tau], W^{1,\infty}(D,R^N))$ with a D bilateral viability condition $< V, n >= 0$ on ∂D Then the flow mapping $T_t(V)$ associated to V maps smoothly the set D onto itself. For any measurable subset $\Omega \subset D$ we defined the convected (or perturbed) domain $\Omega_t = T_t(V)(\Omega)$ whose characteristic function $\zeta(t) = \zeta_\Omega o T_t^{-1}(V)$ solves in distribution sense the classical convection equation 13. Given initial datum ϕ_0 and ψ_0 in $L^2(\Omega)$ and right hand sides f and g in $L^1(0,\tau,L^2(D))$ we consider the following problems

$$\phi(0) = \phi_0,\ \frac{\partial}{\partial t}\phi\, +\, \nabla\phi.V = f \tag{11}$$

$$\psi(0) = \psi_0,\ \frac{\partial}{\partial t}\psi\, +\, div(\psi\, V)\ = g \tag{12}$$

Proposition 4.1 *Assume $V \in L^2(0,\tau,L^2(D,R^N))$ with $div \in L^2(0,\tau,L^2(D))$.*
Then,
if $(divV)^+ \in L^1(0,\tau,L^\infty(D))$ then there exists a solution
$\phi \in L^\infty(0,\tau,L^2(D)) \cap H^1(0,\tau,H^{-1}(D)) \subset C^0([0,\tau],H^{-1/2}(D))$,
if $(divV)^- \in L^1(0,\tau,L^\infty(D))$ then there exists a solution
$\psi \in L^\infty(0,\tau,L^2(D)) \cap H^1(0,\tau,H^{-1}(D)) \subset C^0([0,\tau],H^{-1/2}(D))$

The immediat idea would be to consider $divV \in L^1(0,\tau,L^\infty(D))$.Then both problems have solutions. They are, formely, adjoints problems one an other then we could be tempted to conclude for uniquiness to both problems. Now of course that argument does not works as one of the two solutions ϕ or ψ should be smooth in order to be "put in duality". The second argument would be a renormalisation like consideration: if ϕ was the unique solution associated with $f = 0$, then ϕ^2 would be also a solution by classical chain rule so that we would get $\phi = \phi^2$. Indeed that chain rule would need extra regularity for that solution ϕ. The conclusion is that under previous poor regularity on V we will not get existence nor uniquiness for shape convection problem 13:

$$\zeta(0) = \zeta_{\Omega_0},\ \frac{\partial}{\partial t}\zeta\, +\, \nabla\zeta.V = 0,\ \zeta = \zeta^2 \tag{13}$$

5 Tube by "product space"

To give a sens in 13 to the product $\nabla\zeta.V$ we write it as

$$\nabla\zeta.V = div(\zeta\, V) - \zeta\, divV$$

Then, as soon as $\zeta \in L^\infty((0,\tau)\times D)$, that term makes sense in $L^1(0,\tau,W^{-1,1}(D))$ when V and its divergence $divV$ are in $L^1(0,\tau,L^1(D))$. Through the equation 13, any solution ζ in L^∞ is then in $W^{1,1}(0,\tau,W^{-1,1}(D))$, then in $C^0([0,\tau],W^{-1/2,1}(D))$. Being given a "universe

" bounded domain D, a measurable subset Ω_0 and two real numbers (p,q), $1 < p \leq \infty$, $1 \leq q < \infty$, we consider

$$\mathcal{E}^{p,q} = \{\ V \in L^p(0,\tau,\ L^q(D,R^N))\ s.t.\ divV \in L^p(0,\tau,L^q(D)),\ < V,n >= 0 \quad \text{on}\, \partial D\ \} \tag{14}$$

$$\mathcal{T}^{p,q}_{\Omega_0} = \{\ (V,\zeta) \in \mathcal{E}^{p,q} \times C^0([0,\tau],W^{-1/2,1}(D))\,\text{verifying}\ 13,\ \text{and}\ \zeta\ \in L^1(0,\tau,BV(D))\ \} \tag{15}$$

Notice that in the tubes elements the continuity for the characteristic function ζ is stronger. As $\zeta = \zeta^2$ we get the continuity in $L^p(D)$ for all finite p :

Proposition 5.1 *Let $\zeta \in C^0([0,\tau],W^{-1/2,1}(D))$ and $\zeta = \zeta^2$ then , for all $p,\ 1 \leq p < \infty,\ \ \zeta \in C^0([0,\tau],L^p(D))$*

The proof is done by density of $W^{1/2,1}(D)$ in $L^p(D)$.

The limit case $q = 1$ will be of great interest in that study while the "ideal situation" $p = q = 1$ is still out of the scope of that paper. For several variational situations we shall consider $p = q$ and then we shall write $\mathcal{E}^p$. As soon as ζ verifies $\zeta = \zeta^2$ it can be written as the characteristic function of a measurable subset Q in $(0,\tau) \times D$ (defined up to a $N+1$ zero measure subset), that is $\zeta = \chi_Q$, also for $a.e.t$ we shall write $\zeta(t) = \chi_{\Omega_t}$. The condition $\zeta \in L^1(0,\tau,BV(D))$ is then equivalent to the more usual one

$$\int_0^\tau\ P_D(\Omega_t)\ dt\ \ <\ \infty \tag{16}$$

6 Minimal Speed V^*

We consider the convex set of vector fields V which built the same geometrical tube ζ:

Proposition 6.1 *Assume $q > 1$. For any element $\zeta = (\zeta)^2$ the set*

$$K(\zeta) = \{V \in \mathcal{E}^{p,q} \quad s.t.\ \ (V,\zeta) \in \mathcal{T}^{p,q}_{\Omega_0}\ \} \tag{17}$$

is a (may be empty) closed convex set in $\mathcal{E}^{p,q}$ and then there exist a unique element $V^*_\zeta \in K(\zeta)$ with minimal $L^p(0,\tau,L^q(D))$ norm. In particular for $p = q = 2$ that unique minimal speed vector field is characterised as follows:

$$A(\zeta) = \{\ W \in \mathcal{E}\ s.t.\ \nabla\zeta.W = 0 \quad (\ as\ element\ of\ H^{-1}(D)\ )\,\} \tag{18}$$

$$\forall W \in A(\zeta), \int_0^\tau \int_D (V.W + divV\ divW\,)dxdt = 0 \tag{19}$$

7 $\mathcal{T}_{\Omega_0}^{p,q}$ is weakly closed

We define the topological locally convex linear space $L^\infty(0,\tau,C^0_{comp}(D,R^N))$ as being the inductive limit of the Banach spaces $L^\infty(0,\tau,C^0(K,R^N))$ when K ranges among the compact subsets of D. A sequence g_n converges to zero in that space if there exists a compact set K such that

$$\forall n,\ g_n \in L^\infty(0,\tau,C^0(K,R^N)),\quad ||max_{x\in\bar{D}}\ |g_n(.,x)|\ \ ||_{L^\infty(0,\tau,\ R^N)} \to 0,\quad n\to 0$$

We consider the Banach vector spaces of bounded (vector) measures $\mathcal{M}^1(D,R^N)$ and the Banach space $L^1(0,\tau,\mathcal{M}^1(D,R^N))$ which turns to be in duality with $L^\infty(0,\tau,C^0_{comp}(D,R^N))$ through the obvious bilinear form

$$<\mu,\ g>:=\int_0^1 <\mu(t),\ g(t)>\ dt$$

We recall here the following result of [21]:

Theorem 7.1 *Assume $p>1$, $q>1$. Let $(V_n,\zeta_n)\in\mathcal{T}_{\Omega_0}^{p,q}$ verifying the three following weak convergences (w.c.) :*

$$i)\ \ V_n\,(resp.\,divV_n)\ w.c.\,to\ \ V(resp.\,to\ \ divV\ )\ \ in\ \ L^p(0,\tau,L^q(D,R^N))\ (resp.L^p(0,\tau,L^q(D))\ ) \tag{20}$$

The previous weak convergences refer to $\sigma(L^p(0,\tau,L^q(D)),L^{p^*}(0,\tau,L^{q^*}(D))\ )$ topology.

$$ii)\ \forall g\in L^\infty(0,\tau,C^0_{comp}(D,R^N)),\ \int_0^\tau <\nabla\zeta_n(t),\ g(t)>dt->\int_0^\tau <\nabla\zeta(t),g(t)>dt \tag{21}$$

Then the element (V,ζ) belongs to $\mathcal{T}_{\Omega_0}^{p,q}$.

We investigate now the very important limiting case where p and/or q are equal to 1.

Theorem 7.2 *Assume $p>1$ and $q=1$. Let $(V_n,\zeta_n)\in\mathcal{T}_{\Omega_0}^{p,1}$ and assume that :*

i) V_n (resp. $divV_n$) is bounded in $L^p(0,\tau,L^1(D,R^N))$ (resp. $L^p(0,\tau,L^1(D))$) and V_n (resp. $divV_n$) is weakly convergent in $L^p(0,\tau,\mathcal{M}^1(D,R^N))$ (resp. $L^p(0,\tau,\mathcal{M}^1(D))$) to some bounded measure $\mu\in L^p(0,\tau,\mathcal{M}^1(D,R^N))$ (resp. $div\mu\in L^p(0,\tau,\mathcal{M}^1(D))$)

$$ii)\ \forall g\in L^\infty(0,\tau,C^0_{comp}(D,R^N)),\ \int_0^\tau <\nabla\zeta_n(t),\ g(t)>dt->\int_0^\tau <\nabla\zeta(t),g(t)>dt$$

Then the sequence ζ_n strongly converges in $L^1(0,\tau,L^1(D))$ to some element $\zeta=\zeta^2\in l^\infty((0,\tau\times D))$

The weaker situation concerns the limiting case $p=1,\ q>1$ and the pointwise constraint as follows:

Theorem 7.3 *Let $q \geq 1$, and let $(V_n, \zeta_n) \in \mathcal{T}^{1,q}_{\Omega_0}$ and assume that there exists an element $\theta \in L^1(0,\tau)$ such that*

$$a.e.t,\ 0 < t < \tau,\ \ ||V_n(t)||_{L^1(D,R^N)} + |divV_n|_{L^1(D)} \leq \theta(t) \tag{22}$$

So that V_n (resp. $divV_n$) are bounded in $L^1(\,(0,\tau) \times D,\, R^N)$ (resp. in $L^1(\,(0,\tau)\times D)$) as $||V_n||_{L^1(\,(0,\tau)\times D,R^N)} \leq |\,\theta|_{L^1(0,\tau)}$. Moreover assume:

i) V_n (resp. $divV_n$) is weakly convergent in $\mathcal{M}^1(]0,\tau[\times D, R^N))$ (resp. $\mathcal{M}^1(]0,\tau[\times D))$) to some bounded measure $\mu \in \mathcal{M}^1(]0,\tau[\times D, R^N))$ (resp. $div\mu \in \mathcal{M}^1(]0,\tau[\times D))$)

ii) $\forall g \in L^\infty(0,\tau, C^0_{comp}(D, R^N)),\ \int_0^\tau < \nabla\zeta_n(t),\, g(t) > dt -> \int_0^\tau < \nabla\zeta(t), g(t) > dt$

Then the sequence ζ_n strongly converges in $L^1(0,\tau,L^1(D))$ to some element $\zeta = \zeta^2$.

That results are based on the following extension(already given in [21]) to the space $L^1(0,\tau,BV(D))$ of a result established in fluid dynamic analysis (see [13]) in the classical frame work of reflexive Banach spaces. It concerns the " L^1 compacity" results for sequences f_n of elements in $L^1(0,\tau,BV(D))$ whose (time) derivatives f'_n remain bounded in some weaker spaces. Notice that in all space dimension N we do have

$$BV(D) \subset L^1(D) \subset W^{-1,1}(D)$$

While , if $N \geq 3$, $L^1(D)$ is not a subspace of $H^{-1/2}(D)$, but we have:

$$\forall N,\ N \leq 6,\ BV(D) \subset L^1(D) \subset H^{-2}(D)$$

In the sequel we shall designate by H the space $W^{-1,1}(D)$ or $H^{-2}(D)$ following the context.

Proposition 7.1 *Let f_n be a bounded sequence in $L^1(0,\tau,BV(D))$ and assume one of the two following assumptions:*

i) $\frac{\partial}{\partial t}f_n \in L^1(0,\tau,W^{-1,1}(D))$ (resp. $\frac{\partial}{\partial t}f_n \in L^1(0,\tau,H^{-2}(D))$,) and there exists $\theta \in L^1(0,\tau)$ with

$$a.e.\ t,\quad ||\frac{\partial}{\partial t}f_n(t)||_H \leq \theta(t)$$

Or

ii) $\frac{\partial}{\partial t}f_n \in L^p(0,\tau,H)$, $p > 1$ and $\frac{\partial}{\partial t}f_n$ is bounded in $L^p(0,\tau,W^{-1,1}(D))$ (resp. $L^p(0,\tau,H^{-2}(D))$.

In booth situations ($p = 1$ or $p > 1$), such elements belong to $C^0([0,\tau],W^{-1/2,1}(D))$ (resp. $C^0([0,\tau],H^{-1/2}(D))$).

Finally assume that $f_n(0) = f_0$ is a given element in $BV(D)$.

Then there exists a subsequence strongly convergent in $L^1(0,\tau,L^1(D))$.

Lemma 7.1

$$\forall \eta > 0, \textit{ there exists a constant } c_\eta \textit{ (resp. } \tilde{c}_\eta \textit{) such that } \forall \phi \in BV(D),$$

$$||\phi||_{L^1(D)} \le \eta ||\phi||_{BV(D)} + c_\eta ||\phi||_{W^{-1,1}(D)} (resp. .... + \tilde{c}_\eta \, ||\phi||_{H^{-2}(D)} \;)$$

Proof of the lemma: assume that it is wrong. Then, $\forall \eta > 0$, there exists $\phi_\eta \in BV(D)$ and c_η (resp. $\tilde{c}_\eta$) $\to \infty$ such that

$$||\phi_\eta||_{L^1(D)} \ge \eta ||\phi_\eta||_{BV(D)} + c_\eta ||\phi_\eta||_{W^{-1,1}(D)} \;\; (resp. \; .... + \tilde{c}_\eta \, ||\phi_\eta||_{H^{-2}(D)} \,)$$

We introduce $\psi_\eta = \phi_\eta / ||\phi_\eta||_{BV(D)}$, and we derive:

$$||\psi_\eta||_{L^1(D)} \ge \eta + c_\eta ||\psi_n||_{W^{-1,1}(D)} \ge \eta \;\; (resp. \; ... + \tilde{c}_\eta \, ||\psi_n||_{H^{-2}(D)} \ge \; \eta)$$

But also $||\psi_\eta||_{L^1(D)} \le c||\psi_\eta||_{BV(D)} = c$, for some constant c. Then from the previous inequality : $||\psi_\eta||_{W^{-1,1}(D)} \, (resp. ||\psi_\eta||_{H^{-2}(D)}) \;\to\; 0$. But as $||\psi_\eta||_{BV(D)} = 1$, there exists a subsequence strongly convergent in $L^1(D) \subset W^{-1,1}(D) \cap H^{-2}(D)$, which turns to be strongly convergent to zero. This is a contradiction with $||\psi_\eta||_{L^1(D)} \ge \eta$.

Proof of the proposition: For simplicity we hall denote $H = W^{-1,1}(D) \; (resp. \; H^{-2}(D)\;)$. From the lemma, $\forall r > 0$, there exists a constant c_r such that

$$\forall f \in L^1(0, \tau, BV(D)),$$

$$||f||_{L^1(0,\tau,L^1(D))} \le r||f||_{L^1(0,\tau,BV(D))} + c_r||f||_{L^1(0,\tau,H)}$$

We consider now the terms $f_{n,m} = f_n - f_m$, for $m > n$. With the initial condition $f_{n,m}(0) = 0$.

Given $\epsilon > 0$, by assumption $||f_{n,m}||_{L^1(0,\tau,BV(D))} \le \tau^{1/q} \; ||f_{n,m}||_{L^p(0,\tau,BV(D))} \le \tau^{1/q} \; M$, if we chose r such that $r \, \tau^{1/q} \; M \le 1/2 \, \epsilon$, we shall get ;

$$\boxed{||f_{n,m}||_{L^1(0,\tau,L^1(D))} \le 1/2 \; \epsilon + c_\epsilon \, ||f_{n,m}||_{L^1(0,\tau,H)}}$$

At that point the conclusion will derive if we establish the strong convergence to zero of $f_{n,m}$ in $L^1(0, \tau, H)$.

Now, as $L^1(D) \subset H$, from the boundedness assumption of $||\frac{\partial}{\partial t} f_n||_{L^1(0,\tau,H)}$, we get the boundedness of $f_{n,m}$ in the norm of the space $W^{1,1}(0, \tau, H)$.

As we have the following inclusion: $W^{1,1}(0, \tau, H) \subset C^0([0, \tau], H)$, then the sequence $f_{n,m}$ is bounded in that space of continous functions, we get :

$$\forall t > 0, \; ||f_{n,m}(t)||_H \; \le \;\; M$$

so that by use of Lebesgue dominated convergence theorem it will be sufficient to prove the pointwise convergence of $f_{n,m}(t)$ strongly to zero in H. We shall prove it at any t_0. We have $f_{n,m}(t_0) = a_n + b_n$, with

$$a_n = 1/s \int_{t_0}^{t_0+s} \; f_{n,m}(t)dt, \quad b_n = -1/s \int_{t_0}^{t_0+s} \; (t_0 + s - t) f'_{n,m}(t)dt$$

s is a "free parameter" in the previous identity which simply derives from the "by part integration" of the expression $\int_{t_0}^{t_0+s}(t_0+s-t)f'(s)\,ds$. The advantage of that representation is that the term a_n being a time integral, brings the initial boundness of the sequences f_n and $f_{n,m}$ to the bondness in the usual $BV(D)$ norm for which the usual compacness applies:

$$||a_n||_{BV(D)} \leq 1/s \int_0^\tau ||f_{n,m}(t)||_{BV(D)}dt \leq 2M$$

Then a subsequence converges in $L^1(D)$. Applying that argument for the sequence f_n itself, then to $f_{n,m}$, we derive that $a_n \to 0$ weakly in $BV(D)$, then strongly in$L^1(D)$, then strongly in H as n and $m \to \infty$.

Concerning the term b_n we have to consider booth hypothesis separately:

If $p > 1$ then by Hoelder inequality we get

$$||b_n||_H \leq \int_{t_0}^{t_0+s} ||f'_{n,m}(t)||_H\,dt$$

$$\leq s^{1/q} \int_0^\tau ||f'_{n,m}(t)||_H^p\,dt \leq 2M\,s^{1/q} \tag{23}$$

(with $1/p + 1/q = 1$).

If $\epsilon > 0$ is given we chose s such that $2M\,s^{1/q} \leq \epsilon/2$.

If $p = 1$ we use the function θ and derive :

$$||b_n||_H \leq \int_{t_0}^{t_0+s} ||f'_{n,m}(t)||_H\,dt$$

$$\leq \int_0^s 2\,\theta(t)\,dt \to 0 \;\text{ as }\; s \to 0 \tag{24}$$

Proof of the theorems:

Case $p > 1, q \geq 1$

Let be a sequence $(V_n, \zeta_n) \in \mathcal{T}_{\Omega_0}^{p,q}$ verifying the convergences 20,21. By the Banach Steinhauss uniform bound theorem we get the existence of $M > 0$ such that

$$\int_0^\tau ||\nabla\zeta_n(t)||_{M^1(D)}\,dt \leq M$$

and

$$\int_0^\tau (||V_n(t)||_{L^1(D,R^N)}^p + ||divV_n||_{L^1(D)}^p\,)dt \leq M$$

(that is that $||V_n||_{L^p(0,\tau,L^1(D,R^N))}$ and $||divV_n||_{L^p(0,\tau,L^1(D))}$ are bounded). Also as $\zeta = \zeta^2$ we get $\int_D \zeta_n\,dx \leq\; meas(D)$ so that

$$||\zeta_n||_{L^1(0,\tau,BV(D))} \leq M + \tau\,meas(D)$$

To apply the previous proposition to $f_n = \zeta_n$ we have to verify the assumption concerning f'_n. From the convection equation (that ζ_n element of $\mathcal{T}_{\Omega_0}$ should verifies), we get:

$$\frac{\partial}{\partial t}\zeta_n = -div(\zeta_n\, V_n) + \zeta_n\, divV_n \tag{25}$$

Then $\zeta_n\, V_n$ is bounded in $L^p(0,\tau,L^1(D))$ so that $div(\zeta_n\, V_n)$ is bounded in $L^p(0,\tau,W^{-1,1}(D))$. Then the proposition applies and $\zeta_n \to \zeta = \zeta^2$ in $L^1((0,\tau)\times D)$ (it is well known that the condition $\zeta = \zeta^2$ is closed under L^1 convergence).

Case $p = q = 1,\ \theta$.

again from 25 we get $||\frac{\partial}{\partial t}\zeta_n(t)||_{W^{-1,1}(D)} \leq \theta$,the proposition ii) applies .

In booth cases we have to verify that the limiting element ζ do verifies the convection problem 13.

We weak formulation of 13 is:

$$\forall\, \psi \in C^1([0,\tau]\times \bar{D}),\quad \psi(\tau,.) = 0,$$

$$\int_0^\tau \int_D \zeta_n(-\frac{\partial}{\partial t}\psi\ -\ div(\psi\, V_n)\,)\,dxdt - \int_{\Omega_0} \psi(0,x)dx\ =\ 0$$

in which we pass to the limit as

$$div(\psi\, V_n) = \psi\, divV_n\ +\ \nabla\psi.V_n\ \to\ div(\psi\, V)$$

in $\sigma(L^p(0,\tau,L^1(D)),L^q(0,\tau,L^\infty(D))$ and as $\zeta\ \in\ L^\infty((0,\tau)\times D)\ \subset\ L^q(0,\tau,L^\infty(D))$. Then in the limit we get $V \in K(\zeta)$. In order to get $(V,\zeta) \in \mathcal{T}^{p,BV}_{\Omega_0}$ we have now to prove that $\int_0^\tau\ ||\nabla\zeta(t)||_{M^1(D,R^N)}\ dt\ <\ +\infty$ which will imply that $\zeta \in\ L^1(0,\tau,\ BV(D))$.

D being a bounded domain in R^N with smooth boundary, we consider the locally convex vector space $F = L^\infty(0,\tau,C^0_{comp}(D,R^N))$ as being the inductive limit of the banach spaces $L^\infty(0,\tau,C^0_{comp}(K))$ when K ranges in the family of compact subsets of D. It turns out that we get

$$\int_0^\tau\ ||\nabla\zeta(t)||_{M^1(D,R^N)}\ dt = sup_{\{g\in F,\ a.e.t,\ \forall x\in D,\ ||g(t,x)||\leq 1\}}\quad <\nabla\zeta,\, g>$$

Where the duality brackets $<\ ,\ >$ stands for the duality bilinear form pairing between F and $E = L^1(0,\tau,M^1(D,R^N))$. From Banach Alaglou theorem we get

$$\int_0^\tau\ ||\nabla\zeta(t)||_{M^1(D,R^N)}\ dt\ \leq\ liminf_{\{n\to\infty\}}\ \int_0^\tau\ ||\nabla\zeta_n(t)||_{M^1(D,R^N)}\ dt \tag{26}$$

Corollary 7.1 *The mapping* $(V,\zeta) \longrightarrow \int_0^\tau\ ||\nabla\zeta(t)||_{M^1(D,R^N)}\,dt\ +\ \int_0^\tau\ (\ ||V(t)||^p_{L^1(D,R^N)}\ +\ ||divV||^p_{L^1(D)}\ )\ dt$ *is weakly lower semi continuous on* $\mathcal{T}^{p,q}_{\Omega_0}$.

In the sequel we shall designate , for any element $(V, \zeta = \zeta^2) \in \mathcal{T}_{\Omega_0}^{p,q}$, by χ_{Ω_t} the element $\zeta(t) \in L^p(D), 1 \leq p \leq \infty$.

We have $L^p((0,\tau)\times D) \subset L^p(0,\tau,L^1(D))$ and

$$|\int_D f dx\,|^p \leq meas(D)^{p/q} \int_D |f|^p\,dx, \text{ then } ||f||^p_{L^p(0,\tau,L^1(D))} \leq meas(D)^{p/q}\,||f||^p_{L^p((0,\tau)\times D)}$$

8 Smooth Closure, the set $\mathcal{T}_{\Omega_0}^{p,\infty}$

Of course $\mathcal{T}_{\Omega_0}$ is not empty as it contains tubes associated to smooth fields: for any smooth vector field V, $V \in L^1(0,\tau, Lip(D,R^N))$ we denote by $T_t(V)$ its flow mapping.

For $V \in L^1(0,\tau,W^{1,\infty}(D,R^N))$ and Ω_0 smooth enough, we get

$$P_D(\Omega_t) = \int_{\partial\Omega_0} ||M(DT_t).n_0||ds \leq C\,(\int_0^\tau ||V(t)||_{W^{1,\infty}(D,R^N)}dt\,) \quad P_D(\Omega_0)$$

Then obviously the element $(V, \zeta_{\Omega_0} o T_t^{-1}(V)\,)$ belongs to $\mathcal{T}_{\Omega_0}$. We denote by $\mathcal{T}_{\Omega_0}^{p,\infty}$ the weak closure of such element in $\mathcal{T}_{\Omega_0}^{p,BV}$, that is

$$\forall p, p>1,\ \mathcal{T}_{\Omega_0}^{1,lip} = \{\,(V,\zeta) \in \mathcal{T}_{\Omega_0}\ s.t.\, V \in L^1(0,\tau,W^{1,\infty}(D,R^N))\,\} \subset \mathcal{T}_{\Omega_0}^{p,BV} \tag{27}$$

$$\mathcal{T}_{\Omega_0}^{p,\infty} = cl_{\{\mathcal{T}_{\Omega_0}^{p,BV}\}}\,(\mathcal{T}_{\Omega_0}^{1,lip}) \tag{28}$$

9 The Dual Problem

$$-\frac{\partial}{\partial t}p - \Delta p = 0 \text{ in } Q_V \tag{29}$$

$$p(\tau) = u_\tau \tag{30}$$

$$\frac{\partial}{\partial n_t}p = 0 \quad \text{on the moving boundary}\,\Gamma_t \tag{31}$$

9.0.1 The adjoint weak formulation

is the following one:

$$\forall\,\psi \in C^1([0,\tau]\times R^n) \text{ with } \psi(0) = 0,$$

$$\int_0^\tau \int_{\Omega_t} (p\,\frac{\partial}{\partial t}\psi + \nabla u.\nabla\psi\,)dtdx + \int_0^\tau \int_{\partial\Omega_t} p\,\psi\,v\,d\Gamma_t\,dt = \int_{\Omega_\tau} \psi(\tau)(x)\,dx \tag{32}$$

Setting $P = poT_t(V)$ we get the same equation as 6, but with final condition at $t = \tau$, and the following boundary condition:

$$(DT_t(V)^{-1}.(DT_t(V)^{-1})^*.\nabla P, n> + < DT_t(V)^{-1}.V(t)oT_t(V), n> P$$

$$+ < V(t)oT_t(V), (DT_t(V)^{-1})^*.n > P = 0 \tag{33}$$

For element (V, ζ) in $\mathcal{T}^{1,lip}_{\Omega_0}$ of course we have $\zeta = \zeta_{\Omega_0} oT_t(V)^{-1}$, then the element ζ is uniquely associated to the vector field V.

What is important to understand here is that in the definition 28 the fact that $V \in L^1(0, \tau, W^{1,\infty}(D, R^N))$ is not "important" in the sense that in 27 we could replace the set $\mathcal{T}^{1,lip}_{\Omega_0}$ by the one derived from the vectors fields smoother,

for example $V \in C^0([0, \tau], C^k(D, R^N))$, $k \geq 1$. From "transitivity" of the density property, we will get the same set $\mathcal{T}^{p,\infty}_{\Omega_0}$.

10 Stability of weak relaxed solution

10.0.2 The weak formulation

Given a tube $(V, \zeta) \in \mathcal{T}^{p,\infty}_{\Omega_0}$, we consider the solution u to the weak parabolic problem:

$$\forall\, \psi \in C^1([0, \tau] \times R^n) \text{ with } \psi(\tau) = 0, \int_0^\tau \int_D \zeta(\, -u\, \frac{\partial}{\partial t}\psi \, + \, \mathcal{H}.\nabla\psi\,)dtdx \, = \int_{\Omega_0} \psi(0)(x)\, dx \tag{34}$$

Where

$$\mathcal{H} = \nabla u$$

For the optimal control purpose we introduce the adjoint problem:

10.0.3 The weak relaxed Dual Problem

is the following one:

$$\forall\, \psi \in C^1([0, \tau] \times R^n) \text{ with } \psi(0) = 0,$$

$$\int_0^\tau \int_D \zeta(\,p\, \frac{\partial}{\partial t}\psi \, + \, \mathcal{H}.\nabla\psi\,)dtdx + \int_0^\tau \int_{\Omega_t} p\,\psi\, v\, d\Gamma_t\, dt \, = \int_{\Omega_\tau} \psi(\tau)(x)\, dx \tag{35}$$

10.0.4 solution stability

In order to get $\mathcal{H}^0 = (\nabla u)^0$ in 34, we need the tube to be an open set or at least such property for a.e. t concerning the set Ω_t such that $\zeta(t) = \chi_{\Omega_t}$, $a.e.t.$ A technic to that approch is to "replace" the perimeter by the "density perimeter" $P_\gamma \partial\Omega_t$ that we recall bellow.

Proposition 10.1 *Assume that (ζ_v, V_n) is a sequence of smooth tubes , $\zeta_n = \chi_{Q_n}$, we say that Q_n is builts by smooth speeds vector fields V_n. For each n we classiquely have a solution u_n . Assume that (V_n, ζ_n) converges to (V, ζ) in $\sigma(L^2, L^2) \times L^1$ topology, with $\zeta_n = \zeta_{\Omega_0} o T_t^{-1}(V_n)$. And assume that u_n^0 (the extension by zero) weakly converges in $L^2(0, \tau, D)$ to a limit element u as weel as $(\nabla u_n)^0$ to some element $\mathcal{H}$.. Then $(u, \mathcal{H}) \in L^2(Q)^{N+1}$ and is solution to the problem 34 in Q.*

11 Optimal Control Problem

Let $V \in C^0([0, \tau], C^1(D, R^N))$ with $V.n = 0$ on ∂D and $\zeta = \zeta_0 o T_t(V)^{-1}$

$$j(V, \zeta) = 1/2 \int_0^\tau \int_D \ \zeta(\ (u - U_d)^2 + |\nabla u|^2) dx dt \tag{36}$$

$$j_\sigma(V, \zeta) = j(V, \zeta) + \ \sigma(\int_0^\tau \int_D (||V||^2 + (div V)^2 \,) dx dt + \,||P_\gamma(\partial \Omega_t)||_{BV(0,\tau)} \ \ ) \tag{37}$$

11.1 Optimization Problem

We consider the minimization problem

$$Inf \ \{ \ j(V, \zeta) \ \ | \ \ (V, \zeta) \in \mathcal{T}_\infty \ \} \tag{38}$$

12 Minimizing Sequences

Let (V_n, ζ_n) be a minimizing sequence. The tube Q_n is smooth and the heat equation solution u_n is classiquely defined. Obviously the nul extensions to the cylinder $[0, \tau] \times D$ of both u_n and the gradients ∇u_n are bounded in $L^2([0, \tau] \times D)$. We consider a weakly converging subsequence, still denoted u_n weakly converging to u and ∇u_n weakly converging to some vector field Z. On the other hand, as the $BV(0, \tau)$ norm of $P_\gamma(\partial \Omega_t^n)$ is bounded there exists a subsequence, still denoted Ω^n, such that $P_\gamma(\partial \Omega_t^n)$ converges in $L^1(0, \tau)$ to some integrable function f. Then for almost every t, $P_\gamma(\partial \Omega_t^n) - - > f(t)$ as $n - - > \infty$. As a result $a.e.t$, $P_\gamma(\partial \Omega_t^n) \le M(t)$ then for almost every time t, the open set Ω_t^n converges to some open set Ω_t both in H^c and L^p topologies and $meas(\partial \Omega_t) = 0$ and $P_\gamma(\partial \Omega_t) \le liminf_{n->\infty} P_\gamma(\partial \Omega_t^n) = f(t)$

Let $\phi \in \mathcal{D}([0, \tau] \times D)$ such that $a.e.t$, $\phi(t) \in \mathcal{D}(\Omega_t)$. For $n \ge N_t$ we have $\phi(t) \in \mathcal{D}(\Omega_t^n)$ so that , a.e. t, we have :

$$\int_{\Omega_t^n} \ < \nabla u_n(t), \phi(t) > \ dx = - \int_{\Omega_t^n} \ u_n(t) \, div \phi(t) \ dx$$

Obviously

$$\int_D \ < \nabla u_n(t), \phi(t) > \ dx = \int_{\Omega_t^n} \ < \nabla u_n(t), \phi(t) > \ dx$$

And the same concerning u so that in the limit we get $Z = \nabla u$.

13 Boundedness of the Density Perimeter

13.1 Density Perimeter

Following [7] ,[8] , we consider for any closed set A in D the density perimeter associated to any $\gamma > 0$ by the following.

$$P_\gamma(A) = sup_{\epsilon\in(0,\gamma)} \ [\frac{meas(A^\epsilon)}{2\,\epsilon} \ ] \tag{39}$$

Where A^ϵ is the dilation $A^\epsilon = \cup_{x\in A} B(x,\epsilon)$. We recall some main properties:

$$\text{The mapping } \Omega \ \to P_\gamma(\partial\Omega) \text{ is lower-semi continuous in the } H^c\text{-topology}$$

$$\text{The property } P_\gamma(\partial\Omega) < \infty \text{ implies that meas}(\partial\Omega) = 0 \text{ and } \Omega \ - \ \partial\Omega \text{ is open in } \ D.$$

$$\text{If } \ P_\gamma(\partial\Omega_n) \leq m \text{ and } \Omega_n \text{ converges in the } \ H^c - \text{topology to some open subset}$$

$$\Omega \subset D, \text{then the convergence holds in the } \ L^2(D)\text{topology of the characteristic functions}$$

13.1.1 Clean Open tube

A Clean open tube is a set $\tilde{Q}$ in $]0,\tau[\times D$ such that for $a.e.t$, $\tilde{\Omega}_t = \{x \in D \mid (t,x) \in \tilde{Q}\}$ is an open set in D verifying for almost every t in $(0,\tau)$ the following cleaness property:

$$meas(\partial\tilde{\Omega}_t) = 0, \ \ \tilde{\Omega}_t = \text{ interior of } cl(\tilde{\Omega}_t). \tag{40}$$

Notice that as the previous openess condition holds at almost every time t , the set Q is not necessarelly itself an open subset in $]0,\tau[\times D$. Nethertheless when the field V is smooth the tube $\bigcup_{0<t<\tau} \{t\} \times T_t(V)(\Omega_0)$ is open (resp. open and clean open) when the initial set Ω_0 is open (resp. clean open) in D.

We say that two tubes Q and Q' are equivalent if the characteristic functions are equal as elements in $L^2(0,\tau,L^2(D))$ (i.e. $\xi_Q = \xi_{Q'}$), that is to say that at almost every time t the two sets $\{x \in D \mid (t,x) \in Q\}$ and $\{x \in D \mid (t,x) \in Q'\}$ are the same up to a mesurable subset E of D verifying $meas(E) = 0$.

Lemma 13.1 *Let Q be a mesurable set in $]0,\tau[\times D$, if there exists a clean open tube $\tilde{Q}$ such that $\xi_Q = \xi_{\tilde{Q}}$, then that clean tube is unique.(There exists at most one equivalent clean open tube)*

Proof: assume two such clean tubes $\tilde{Q}$ and $\tilde{Q}'$. Then at a.e. t we have $\tilde{\Omega}_t = \tilde{\Omega}'_t$ up to a mesurable set E_t verifying $meas(E_t) = 0$. As those two open set verify (40) they are equals.

When Ω_0 is a clean open in D and Q is a clean open tube in $]0,\tau[\times D$, with $\xi_Q \in C^0([0,\tau], H^{-1/2}(D))$ and such that there exists a divergence free field V in E such that :

$$\frac{\partial}{\partial t}\xi_Q + \nabla \xi_Q.V = 0, \quad \xi_Q(0) = \xi_{\Omega_0} \tag{41}$$

we say that V builts the tube and we note $Q = Q_V$. Now such field when it exists is not unique. The set of fields that built the clean open tube Q is closed and convex:

$$\mathcal{V}_Q = \{V \in E \quad | \; Q_V = Q\} \tag{42}$$

Lemma 13.2 *If the set $\mathcal{V}_Q$ is non empty, then it is closed and convex in E so it contains a unique element V_Q which minimizes the $L^2(0,\tau,L^2(D))$ norm in that class.*

When the tube Q is built by a smooth field $V \in L^1(0,\tau,W_0^{1,\infty}(D,R^N))$, that is $Q = Q_V$ (i.e. $\Omega_t = \{x \in D \mid (t,x) \in Q\} = T_t(V)(\Omega_0)$), obviously the convex set $\mathcal{V}_{Q_V}$ is non empty as $V \in \mathcal{V}_{Q_V}$. But in general V_{Q_V}, the minimum L^2-norm element in $\mathcal{V}_{Q_V}$, is different from V.

We describe now a construction of clean open tubes for which the set $\mathcal{V}_Q$ is non empty.

13.1.2 The "parabolic" situation

We turn to the situation of dynamical domain. One could think to use the time-space perimeter as it was considered in ([12]). For any smooth free divergence vector field, $V \in C^0([0,\tau], W_0^{1,\infty}(D,R^N))$, we consider,

$$\Theta_\gamma(V,\Omega_0) \;=\; Min\;\{\int_0^\tau (\frac{\partial}{\partial t}\mu)^2\; dt \,|\, \mu \in \mathcal{M}_\gamma(V,\Omega_0)\;\} \tag{43}$$

Where

$$\mathcal{M}_\gamma(V,\Omega_0) = \{\;\mu \in H^1(0,\tau),\, P_\gamma(\partial\Omega_t(V)) \le \mu(t)\; a.e.t, \quad \mu(0) \le (1+\gamma)P_\gamma(\partial\Omega_0)\,\}$$

In general that set is non empty. When that set is empty we put $\Theta_\gamma(V,\Omega_0) = +\infty$. Notice that even when the mapping $p = (t \longrightarrow P_\gamma(\Omega_t(V))\;)$ is an element of $H^1(0,\tau)$ (then $p \in \mathcal{M}_\gamma(V,\Omega_0)\;\}$), we may have: $\Theta(V,\Omega_0) < ||p'||^2_{L^2(0,\tau)}$ as the minimizer will escape to possible variation of the function p.

Proposition 13.1 *For any smooth free divergence field $V \in C^0([0,\tau], W_0^{1,\infty}(D,R^N))$, we have:*

$$P_\gamma(\partial\Omega_t(V)) \le 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}\;\;\Theta(V,\Omega_0)^{1/2} \tag{44}$$

proof. As

$$P_\gamma(\partial\Omega_t(V)) \le \mu(t) \le 2P_\gamma(\partial\Omega_0) + \int_0^\tau \frac{\partial}{\partial t}\mu(t)dt$$

Then

$$P_\gamma(\partial\Omega_t(V_n)) \leq \mu(t) \leq 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}(\int_0^\tau (\frac{\partial}{\partial t}\mu(t)\,)^2\,dt\,)^{1/2}$$

as μ is chosen being the minimizer element associated with V, the estimation is proved.

Proposition 13.2 *Let $V_n \in C^0([0,\tau], W_0^{1,\infty}(D,R^N))$, with the following convergence:*

$$V_n \longrightarrow V \;\; in \;\; L^2((0,\tau)\times D, R^N)$$

and the uniform boundedness :

$$\exists M > 0, \quad \Theta(V_n,\Omega_0) \leq M$$

Then

$$\Theta(V,\Omega_0) \leq \; liminf \;\, \Theta(V_n,\Omega_0)$$

Proof. With the boundedness assumption :

$$P_\gamma(\partial\Omega_t(V_n)) \leq C = 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}\;\; M^{1/2}$$

Let μ_n be the unique minimizer in $H^1(0,\tau)$ associated with $\Theta(V_n,\Omega_0)$. There exists a subsequence, still denoted μ_n , which weakly converges to an element $\mu \in H^1(0,\tau)$. That convergence holds strongly in $L^2(0,\tau)$, then almost every where. By definition we have

$$P_\gamma(\partial\Omega_t(V_n)) \leq \;\mu_n \;a.e.$$

Then in the limit:

$$P_\gamma(\partial\Omega_t(V)) \leq \; liminf P_\gamma(\partial\Omega_t(V_n)) \leq \;\mu(t), \;\; a.e.t$$

Also the square of the norm being weakly lower semi continuous we have

$$\int_0^\tau (\frac{\partial}{\partial t}\mu(t))^2 dt \;\;\leq \;\; liminf \; \int_0^\tau (\frac{\partial}{\partial t}\mu_n(t))^2 dt$$

that leads to

$$\Theta(V,\Omega_0) \leq \int_0^\tau (\frac{\partial}{\partial t}\mu(t))^2 dt \leq \; liminf \;\, \Theta(V_n,\Omega_0)$$

14 The Dual Problem

$$-\frac{\partial}{\partial t}p \;-\; \Delta p \;=\; 0 \text{ in } Q_V \tag{45}$$

$$p(\tau) = u_\tau \tag{46}$$

$$\frac{\partial}{\partial n_t}p \;=\; 0 \quad \text{on the moving boundary}\,\Gamma_t \tag{47}$$

14.0.3 The adjoint weak formulation

is the following one:

$$\forall\,\psi \in C^1([0,\tau]\times R^n) \text{ with } \psi(0)=0,$$

$$\int_0^\tau \int_{\Omega_t} (p\,\frac{\partial}{\partial t}\psi\, +\, \nabla u.\nabla\psi\,)dtdx + \int_0^\tau \int_{\partial\Omega_t} p\,\psi\,v\,d\Gamma_t\,dt \;=\; \int_{\Omega_\tau} \psi(\tau)(x)\,dx \qquad (48)$$

Setting $P = poT_t(V)$ we get the same equation as 6, but with final condition at $t=\tau$, and the following boundary condition:

$$(DT_t(V)^{-1}.(DT_t(V)^{-1})^*.\nabla P, n> + < DT_t(V)^{-1}.V(t)oT_t(V), n>\ P$$

$$+ < V(t)oT_t(V), (DT_t(V)^{-1})^*.n > P = 0 \qquad (49)$$

15 Metric and Geodesic

Before giving the definition of the tube metric in some subfamilly of subsets $\Omega \subset D$ with finite perimeter let us analyse sevral "pseudometrics" for which the third metric axiom (triangle inequality). The question adressed here is to showthat with distance definition involving only the integral of th perimeter of a tube we are obliged, in order to verify the first axiom, to introduce a correction term. That correction term will always empeche to reach the triangle inequality.

15.1 First pseudometric d_p, $p > 1$

We consider the final time $\tau = 1$, and for any sets $\Omega_1$and Ω_2 with bounded perimeter in D we consider the set (may be empty):

$$\mathcal{T}^p_{\Omega_1,\Omega_2} = \{\ (V, \zeta = \zeta^2) \in \mathcal{T}^{p,1}_{\Omega_1} \text{ such that } \ \zeta(0) = \chi_{\Omega_1},\ \zeta(1) = \chi_{\Omega_2}\ \}$$

We have $\zeta \in C^0([0,\,1],\, W^{-1/2,1}(D))$ so that we consider the sets Ω_1 and Ω_2 defined up to a set of measure zero by there characteristic function, $\chi_{\Omega_1} = \zeta(0)$ $\chi_{\Omega_2} = \zeta(1)$. The elements $\chi_{\Omega_i} \in L^1(D) \subset W^{-1/2,1}(D)$ and $\mathcal{T}^{p,1}_{\Omega_1,\Omega_2} \subset C^0([0,\,1],\, W^{-1/2,1}(D))$ we set

$$d(\Omega_1,\Omega_2) = INF\{\ \int_0^1\ |P_D(\Omega_t(V)) - \frac{P_D(\Omega_1)+P_D(\Omega_2)}{2}|\,dt + (\int_0^1 \int_D (\ ||V(t)||^p + |divV(t)|^p\ )\,dt\,)^{1/p}\} \qquad (50)$$

Infimum being peformed over the tubes $\mathcal{T}_{\Omega_1,\Omega_2}p, 1$.

It is obviously symmetrical and if $d(\Omega_1,\Omega_2) = 0$ we get $V = 0$ then $\Omega_1 = \Omega_2$ which is compatible with $\forall t, 0 < t < 1$, $\Omega_t(V) = \Omega_1 = \Omega_2$, then $P(\Omega_t) = \frac{P_D(\Omega_1)+P_D(\Omega_2)}{2}$. The delicate point is the triangle inequality.The idea is based of the monotonicity of the perimeter $P_D(\Omega_t)$ (with respect to t) for optimal tubes. Assume $P(\Omega_2) > P_D(\Omega_1)$, then

$$\frac{|P_D(\Omega_1) - P_D(\Omega_2)|}{2} \leq \int_0^1 |P(\Omega_t) \; - \; \frac{P_D(\Omega_1) + P_D(\Omega_2)}{2} \; | \, dt \tag{51}$$

Assume that (ξ^1, V^1) is an optimal tube for $d(\Omega_1, \Omega_2)$ while (ξ^2, V^2) is optimal tube for $d(\Omega_2, \Omega_3)$. We consider the vector field

$$V(t,.) = 2V_1(2t) \text{if } 0 < t < 1/2, \;\; V(t,.) = 2V^2(2t-1) \text{ if } 1/2 < t < 1$$

with $\xi(t) = \xi^1(2t), \; 0 < t < 1/2, \;\; \xi = \xi^2(2t-1) \; 1/2 < t < 1$ we verify easily that (ξ, V) is a tube connecting Ω_1 to Ω_3 so that

$$d(\Omega_1, \Omega_3) \leq \int_0^1 \; |P_D(\Omega_t(V)) - \frac{P_D(\Omega_1) + P_D(\Omega_3)}{2}| \, dt + (\int_0^1 \int_D (\; ||V(t)||^p + |divV(t)|^p \;) \, dtdx \,)^{1/p}$$

$$= \int_0^{1/2} \; |P_D(\Omega_t(V)) - \frac{P_D(\Omega_1) + P_D(\Omega_3)}{2}| \, dt \; + (\int_0^{1/2} \int_D (\; ||V(t)||^p \; + \; |divV(t)|^p \,) \, dtdx \,)^{1/p}$$

$$+ \int_{1/2}^1 \; |P_D(\Omega_t(V)) - \frac{P_D(\Omega_1) + P_D(\Omega_3)}{2}| \, dt$$

$$+ \;\; (\int_{1/2}^1 \int_D (\; ||V(t)||^p \; + \; |divV(t)|^p \,) \, dtdx \,)^{1/p}$$

$$= 1/2 \int_0^1 \; |P_D(\Omega_t(V^1)) - \frac{P_D(\Omega_1) + P_D(\Omega_2)}{2} + \frac{P_D(\Omega_2) - P_D(\Omega_3)}{2}| \, dt$$

$$+ \, 2^{\frac{p-1}{p}} \; (\int_0^1 \int_D (\; ||V^1(t)||^p \; + \; |divV^1(t)|^p \,) \, dtdx \,)^{1/p}$$

$$+1/2 \int_0^1 \; |P_D(\Omega_t(V^2)) - \frac{P_D(\Omega_2) + P_D(\Omega_3)}{2} + \frac{P_d(\Omega_2) - P_D(\Omega_1)}{2}| \, dt$$

$$+2^{\frac{p-1}{p}} \, (\int_0^1 \int_D (\; ||V^2(t)||^p \; + \; |divV^2(t)|^p \,) \, dtdx \,)^{1/p}$$

$$\leq 1/2 \int_0^1 \; |P_D(\Omega_t(V^1)) - \frac{P_D(\Omega_1) + P_D(\Omega_2)}{2}| dt + |\frac{P_D(\Omega_2) - P_D(\Omega_3)}{2}|$$

$$+2^{\frac{p-1}{p}} \, (\int_0^1 \int_D (\; ||V^1(t)||^p \; + \; |divV^1(t)|^p \,) \, dtdx \,)^{1/p}$$

$$+1/2 \int_0^1 \; |P_D(\Omega_t(V^2)) - \frac{P_D(\Omega_2) + P_D(\Omega_3)}{2}| dt + |\frac{P_d(\Omega_2) - P_D(\Omega_1)}{2}|$$

$$+2^{\frac{p-1}{p}} \, (\int_0^1 \int_D (\; ||V^2(t)||^p \; + \; |divV^2(t)|^p \,) \, dtdx \,)^{1/p}$$

Then $\boxed{d(\Omega_1, \Omega_3) \leq \; 2^{\frac{p-1}{p}} \; (\, d(\Omega_1, \Omega_2) + d(\Omega_2, \Omega_3) \;) \;\; + \;\; |\frac{P_d(\Omega_2) - P_D(\Omega_1)}{2}| + |\frac{P_D(\Omega_2) - P_D(\Omega_3)}{2}|}$

15.2 Second pseudometric $\bar{d}_p$, $p > 1$

In the very definition of the metric d_p we replace d the midle value of the perimeter by its average along the tube as follows:

$$\bar{d}_p(\Omega_1, \Omega_2) = \int_0^1 |P_D(\Omega_t) - \bar{P}_D(\Omega_t)| + ||V||^p_{L^p((0,\tau)\times D)}$$

where $\bar{P}_D(\Omega_t) = \int_0^1 P_D(\Omega_t)\, dt$ (This term "replaces" the term $\frac{P_D(\Omega_1)+P_D(\Omega_2)}{2}$ in the previous metric d_p). Again if $\bar{d}_p(\Omega_1, \Omega_2) = 0$ we get $V = 0$ then ζ is constant with respect to time, that is that the tube is a cylinder, then $\forall t,\ P_D(\Omega_t) = \bar{P}_D$ and $\Omega_1 = \Omega_2$ we have

$$\int_0^1 ||\nabla\zeta(t)||dt = \int_0^{1/2} ||\nabla\zeta^1(2t)||dt + \int_{1/2} ||\nabla\zeta^2(2t-1)||\, dt$$

That is:

$$\bar{P}_D(\Omega_t) \ = 1/2(\ \bar{P}^1_D \ + \ \bar{P}^2_D\ ) \ \text{ with } \bar{P}^i_D = \int_0^1 ||\zeta^i(t)||dt$$

we get

$$\bar{d}_p(\Omega_1, \Omega_3) \le \int_0^1 |\ ||\nabla\zeta(t)|| - \int_0^1 ||\nabla\zeta(s)||ds\ |dt + \ ||V||$$

Where, as previously,

$$V(t) = V^1(2t),\ 0 < t < 1/2, \quad V(t) = V^2(2t-1),\ 1/2 < t < 1$$

Similarly we get

$$\begin{aligned}\bar{d}_p(\Omega_1, \Omega_3) \le 1/2\ (\int_0^1 |\ ||\nabla\zeta^1|| - \int_0^1 ||\nabla\zeta^1||\, ds \quad |\ dt + \int_0^1 |\ ||\nabla\zeta^2|| - \int_0^1 ||\nabla\zeta^2||ds\,|\,dt\ )\\ + 1/2(\int_0^1 |\ ||\nabla\zeta^1|| - ||\nabla\zeta^2||\ |dt\\ + \frac{1}{2}\ (||V^1||\ +\ ||V^2||\ )\end{aligned}$$

that is

Proposition 15.1 *Let $p > 1$*

$$\bar{d}_p(\Omega_1, \Omega_2) = 0 \ \textit{ if and only if } \ \chi_{\Omega_1} = \chi_{\Omega_2}$$
$$\bar{d}_p(\Omega_1, \Omega_2) = \bar{d}(\Omega_2, \Omega_1)$$

$$\boxed{\bar{d}_p(\Omega_1, \Omega_3) \le\ 2^{p-1}\,(\,\bar{d}(\Omega_1, \Omega_2) + \bar{d}_p(\Omega_2, \Omega_1)\,)\ + 1/2\ \int_0^1\ |\ \ ||\nabla\zeta^1||\ -\ ||\nabla\zeta^2||\ |dt}$$

.

15.3 Third pseudometric

We consider the family of tubes such that the mapping $p = (t \longrightarrow P_D(\Omega_t) := ||\nabla\zeta(t)||_{M^1(D;R^N)})$ is dominated by a $H^1(0,\tau)$ function μ.

For any tube (ζ, V) we consider the (may be empty) closed convex set

$$M(\zeta,V) = \{\ \mu \in H^1(0,1),\ s.t.\ p(t) \leq \mu(t),\ a.e.\, t\ ,\ \mu(0) = P_D(\Omega_1),\ \mu(1) = P_D(\Omega_2)\ \}$$

and the family of tubes for which that set is non empty, that is:

$$\tilde{\mathcal{T}}_{\Omega_1,\Omega_2} = \{(\zeta,V) \in \mathcal{T}_{\Omega_1,\Omega_2}\ \ s.t.\ \exists\mu \in M(\zeta,V)\ \ \}$$

Obviously, for any element $(\zeta,V) \in \tilde{\mathcal{T}}_{\Omega_1,\Omega_2}$, the previous function $p(t)$ belongs to $L^\infty(0,\tau)$. is then

$$\tilde{d}(\Omega_1,\Omega_2) := INF_{(\zeta,V)\in\tilde{\mathcal{T}}_{\Omega_1,\Omega_2},\mu\in M(\zeta,V)} \int_0^1 (\frac{\partial}{\partial t}\mu(t)\ )^2\, dt + \int_0^1 (\ ||V(t)||_{L^q(D,R^N)} + |divV|_{L^q(D)}\,)\, dt$$

That Infimum being considered among the admissible tubes $(\zeta,V) \in \tilde{\mathcal{T}}_{\Omega_1,\Omega_2}$ and all $\mu \in M(\zeta,V)$. Let

$$\delta(\zeta,V) := MIN_{\mu\in M(\zeta,V)} \int_0^1 (\frac{\partial}{\partial t}\mu(t)\ )^2\, dt$$

So that

$$\tilde{d}(\Omega_1,\Omega_2) = INF_{(\zeta,V)\in\tilde{\mathcal{T}}_{\Omega_1,\Omega_2}} \left(\ \delta(\zeta,V)\ +\ \int_0^1 (\,||V(t)||_{L^q(D,R^N)} + |divV|_{L^q(D)}\,)\, dt\ \ \right) \quad (52)$$

$$d_q(\Omega_1,\Omega_2) = MIN \quad \int_0^1 (\ \int_D (||V(t,x)||^q + |divV(t,x)|^q)dx\ +\ p(t)\ )dt \quad (53)$$

Proposition 15.2 *The infimum is reached in 52.*

Proof: let (ζ_n, V_n) be a minimizing sequence. Let μ_n be the minimizer in $\delta(\zeta_n,V_n)$, then there exist a constant M such that $|p_n|_{L^\infty(0,\tau)} \leq M$. From the theorem 1 we get the strong convergence $\zeta_n \to \zeta = \zeta^2$ in $L^2(0,\tau,L^2(D))$, and V_n $(resp.\, divV_n) \rightharpoonup V$ $(resp.\, divV)$ weakly in $L^2(D,R^N)$ $(resp.\, L^2(D))$. In the limit we get $(\zeta,V) \in \mathcal{T}_{\Omega_1,\Omega_2}$. Moreover $p(t) \leq liminf\, p_n(t)$ then $M(\zeta,V)$ is non empty and $(\zeta,V) \in \tilde{\mathcal{T}}_{\Omega_1,\Omega_2}$.

15.4 Shape Metric by Tube geodesic

In fact here we need not the perimeter to be bounded as it was necessary previously when using the density perimeter. Then in stead of introducing the previous terms we can directely introduce in the distance expression the following additive term :

$$\boxed{\int_0^1 p(t)\,dt, \quad \text{where} \;\; p(t) = ||\nabla\zeta(t)||_{M^1(D)}}$$

So the "ideal" distance formulation would be

$$d(\Omega_1,\Omega_2) = INF \int_0^1 (\int_D (||V(t,x)|| + |divV(t,x)|)dx \;+\; p(t)| \;)dt$$

Now that kind of approach makes sens in the context of theorem 7.3as follows : introducing any element $\theta \in L^1(0,1)$ and taking the previous minimum over vector field in L^1 whose L^1 norm is dominated by θ which leads to a distance function d_θ. If we want to escape to that "θ behavior", up to now, we need some power $q > 1$ in the Speed norme term in order to preserve in the limit the fundamental property of tube $\zeta = \zeta^2$

Definition 15.1 *Let $p \geq 1$ and $\theta \in L^1((0,1)\times D)$ be given. We set*

$$\mathcal{T}^{p,\theta}_{\Omega_1,\Omega_2} = \{\, (\zeta,V) \in \mathcal{T}^{p}_{\Omega_1,\Omega_2} \; s.t. \;\; ||v(t,x)|| + |divV(t,x)| \;\leq\; \theta(t,x) \;\, a.e.(t,x) \in (0,1)\times D\} \tag{54}$$

For any subset $\Omega \subset D$ with finite perimeter in D, we consider the family of subset Ω' in D with finite perimeter and being "(p,θ)-connected to Ω" in the following sens:

$$\mathcal{T}^{p,\theta}_{\Omega} = \{\;\Omega' \;\; s.t.\; \exists(\zeta,V) \in \mathcal{T}^{p,\theta}_{\Omega,\Omega'} \;\;\}. \tag{55}$$

For any couple $(\Omega_1,\Omega_2) \in \mathcal{T}^{p,\theta}_{\Omega}$ we set:

$$d(\Omega_1,\Omega_2) = MIN_{(\zeta,V)\in\mathcal{T}^{1,\theta}_{\Omega_1,\Omega_2}} \int_0^1 \int_D (||V(t,x)|| + |divV(t,x)|)dx\; dt + \;\; ||p(t)||_{BV(0,1)} \tag{56}$$

Notice that the definition is made possible as $\Omega_i \in \mathcal{T}^{p,\theta}_{\Omega}$, $i = 1,2$, implies that the set $\mathcal{T}^{p,\theta}_{\Omega_1,\Omega_2}$ is non empty. From the same argument then in the previous proposition proof we see that the minimum is reached. Notice also that the two sets Ω_1 and Ω_2 in $\mathcal{T}^{p,\theta}_{\Omega}$ have not necessarely the same topology. The "(p,θ)-connection" dœs not implies any existence of homeomorphisism between these two sets, but only the existence of a tube (ζ,V), with $\zeta \in C^0([0,1],H^{-1/2}(D))$ and $\zeta(0)\chi_{\Omega_1}$, $\zeta(1) = \chi_{\Omega_2}$.

Theorem 15.1 *Let be given a subset Ω of D with finite perimeter in $D$and $\theta \in L^\infty((0,1)\times D)$. Then d is a metric on the family $\mathcal{T}^{1,\theta}_{\Omega}$.*

Proof: From the previous developments on the pseudometrics, the only point is to verify the third metric axiom (which was missing in the previous pseudo metric). Assume be given three sets Ω_i with finite perimeter in D. Let (ζ_1,V_1) a tube connecting Ω_1 to Ω_2 and (ζ_2,V_2) connecting Ω_2 to Ω_3 and realizing the minimun in the associated distances expressions. As previously we built the tube $\zeta(t) = \zeta_1(2t)$ if $0 < t < 1/2$, $= \zeta_2(2t-1)$ if $1/2 < t < 1$ and the same for the field $V(t) = 2V_1(2t)$ or $= 2V_2(2t-1)$.

Definition 15.2 *The tubes (ζ,V) achiving the minimum in that distance definition 56 are called the $\mathcal{T}$ube geodesic .*

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